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Self-organized criticality in a simple model of neurons based on small-world networks

Min Lin*
Department of Physics, Nankai University, Tianjin 300071, People's Republic of China

TianLun Chen†
CCAST (World Laboratories), P. O. Box 8730, Beijing 100080, People's Republic of China
and Department of Physics, Nankai University, Tianjin 300071, People's Republic of China
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A simple model for a set of interacting idealized neurons with small-world structure is introduced. The basic elements of the model are endowed with the main features of a neuron function. We find that our model displays power-law behavior of avalanche sizes and generates long-range temporal correlations and $1/f$ noise. More importantly, we find there are different avalanche dynamical behaviors for different ϕ, the density of short paths in the network.

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I. INTRODUCTION

In 1987, Bak, Tang, and Wiesenfeld introduced the concept of "self-organized criticality" (SOC) [1]. It is shown that extended nonequilibrium systems can organize into a scale-invariant critical state spontaneously, without fine tuning of a control parameter. This critical state is characterized by a power-law distribution of avalanche sizes, which is regarded as a "fingerprint" for SOC.

The brain, which possesses about $10^{10}-10^{12}$ neurons, is one of the most complex systems. Now evidence for some aspects of scale invariance has been found in the central nervous system [2]. Some scientists stated that the brain might be operating at, or near, a critical state [3]. The mechanism of the SOC process in the brain is an interesting research area.

It is well known that network connectivity in the cortex and other brain regions is mainly local, with relatively sparse long-distance projections. From a neuron-biological point of view, unlike for fully connected artificial neural networks, plausible associative memories must have sparse connectivity, reflecting the situation in the cortex and hippocampus [4]. Recently, Watts and Strogatz [5] studied a class of networks that can be tuned from regular to random states by varying a single parameter. They called this class of networks "small-world" networks. Small-world networks can be highly clustered, like a regular lattice, yet have a small characteristic path length, like random graphs. The key role of a few long-range connections (shortcuts) in the small-world network is to decrease the length of the characteristic path. It suggests that small-world phenomena might be common in sparse networks with many nodes. Indeed, it has been proven that the nervous system of the nematode worm *C. elegans* shows small-world properties [5]. Small-world networks of coupled phase oscillators are optimal for producing synchronization. The results may be relevant to the observed synchronization of widely separated neurons in the cat visual cortex [6]. Watts and Strogatz proposed that the brain has a small-world architecture [5].

A simple model for brain functioning that presents a self-organized state was introduced in Ref. [7]. In this paper, we discuss the model in Ref. [7] based on small-world networks. Our aim is to investigate the influence of network topology on dynamical behaviors. The network topology is completely regular in Ref. [7], while our network topology has a small-world structure. We find our model displays power-law behavior of avalanche size and generates long-range temporal correlations and $1/f$ noise in the activity of neural populations. More importantly, we find that there are different avalanche dynamical behaviors for different rewiring probability ϕ, the density of long-range connections in the small-world networks.

II. THE MODEL

Our model is a one-dimensional and L-node lattice system based on a small-world network. Each node represents a neuron; a connection between two nodes represents a synapse. According to the neuron-dynamical picture of the brain, the dynamics of neurons and synapses can be described as follows.

The axon, by order of its neuron (presynaptic neuron), generates a signal with the form of an action potential. The amplitude of the spike is of the order of tens of millivolts. The signal is transferred by the synapse to the soma of another neuron (postsynaptic neuron), where the inputs from all the presynaptic neurons connected to it are summed. On average each neuron is connected to 10^3-10^4 other neurons. The amplitude of the input signals at the soma is about 1 mV. These inputs may be either excitatory, hence favoring the likelihood of the appearance of a spike to be transmitted to its postsynaptic neurons, or inhibitory, reducing the likelihood of firing. If the sum of excitatory signals exceeds some threshold, the probability for the emission of a spike, which is the manifestation of the instability, becomes significant. This threshold is tens of millivolts high, and hence a

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*Electronic address: linminmin@eyou.com
†Electronic address: chenzh@nankai.edu.cn

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great number of excitatory inputs make the corresponding neuron unstable, leading to the firing and allowing a spike to occur. After the release of a spike, the neurons require a period of time to recover. During the 1–2 ms following the emission of a spike, no matter how large the excitatory input may be, the neuron is unable to emit a second spike. This period is called the absolute refractory period of the neuron. On the other hand, the brain as a whole is a system capable of automodifications [7].

In our model, we intend to capture the main characteristics of all that was stated above. First, we provide a detailed description about the construction of small-world networks. The small-world networks are generated by using the procedure outlined by Watts and Strogatz [5].

(1) Start with a one-dimensional regular lattice with L sites. The periodic boundary condition is used in the lattice. Each lattice site is connected to its $2k$ nearest neighbors by a bond (so that for L interacting sites, or neurons, we have Lk bonds).

(2) Randomly choose two sites of the lattice and place a bond between them. Self-connections and duplicate links are excluded. Then one of the smaller bonds going to a neighbor site of one of the end points of one long bond is moved.

(3) Repeat step 2 until the number of bonds "rewired" is the fraction ϕ of all bonds of the original lattice, i.e., ϕkL.

Steps 2 and 3 mean that we rewired with probability ϕ one long-range bond for each connection on the original lattice. Two connected sites are indicated as "the nearest neighbor." The situation $\phi=0$ corresponds to the simple regular lattice and large ϕ corresponds to the random graph.

The dynamical process of our model is defined and simulated as follows.

(a) L neurons are distributed on a small-world network. With each site we associate a random barrier B_i, uniformly distributed between 0 and 1.

(b) At each time step, the lowest barrier is found and the corresponding neuron is fired by assigning a new random number between 0 and 1 to the barrier, and by assigning to all its nearest neighbors new random numbers between 0 and 1.

(c) Last, the site that suffers a change in its barrier as a consequence of firing itself is prohibited from firing again during a period of time T_r (the refractory period). Its neighbors are free to fire at any moment if they satisfy the condition of being the lowest. If after a certain time interval $t < T_r$ a nearest neighbor is fired, the barrier of the temporarily "frozen" neuron is also changed but it continues to be prohibited to produce a spike until a time $t > T_r$ has elapsed.

In our model we do not attempt to give a detailed description of the elements of the brain. Instead, we represent each neuron with a barrier (a real number between 0 and 1) that characterizes its instantaneous probability of releasing a spike, which is the measure of the instability of the neuron. The barrier height of a given neuron separates its current state (characterized by its local probability of firing) from other more stable states. Low-barrier neurons are easy to fire and high-barrier neurons are difficult to fire [7]. When a neuron fires, it changes the instability of the nearest neighbors. A neuron with a low probability of firing (high barrier) has a high chance to fire in subsequent time steps by a firing coming from a related neuron, causing a reduction of the barrier. An inhibitory stimulus of a related neuron could cause an increase in the barrier height, retarding any possible firing action. The modification of the barrier can be thought of as either the result of the release of a spike by its own neuron or the consequence of a received signal that changes the stability of the neuron. At the end of this paper we will also find that the system we are modeling is not only self-organized but also critical.

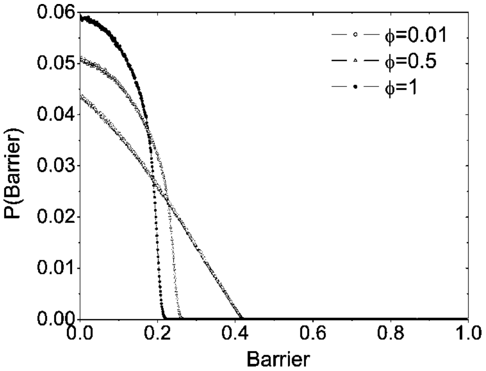

FIG. 1. The distribution of the minimum barriers B_{min} with different ϕ. The system size used is $L=1000$, $k=2$, and the absolute refractory period $T_r=1$.

III. SIMULATION RESULTS

Here we use a system of the size $L=1000$ with $k=2$. This means that the ranges of interactions go beyond the first-neighbor barriers. Then we change ϕ; our aim is to investigate avalanche dynamical behaviors for different ϕ.

If we begin the system with an arbitrary distribution of barrier heights, subsequent firing activities would be completely uncorrelated but, as time goes on, it would become more and more likely that the nearest neighbors are next to fire (respecting the constraint of the refractory period). After a transient, the system reaches a highly correlated stationary state. All the minimum barriers $B_i(t)$ are less than what is called the "self-organized threshold" B_c in Ref. [7]. A self-organized threshold also exists in our model. From Fig. 1, for a certain ϕ, we can see that the distribution of the lower barriers in the critical state vanishes at and above the corresponding $B_c(\phi)$. We call the "self-organized threshold" for a certain ϕ in our model $B_c(\phi)$. We find that $B_c(\phi)$ decreases with the increment of ϕ. It can be explained that with the increment of ϕ, the number of long-range connections (ϕkL) increases and the number of links of some particular sites exceeds 4. In this case, the scopes of the particular sites, which connect the nearest neighbors, enlarge. Thus the ranges of the particular sites' local interaction are extended and the signal of the fired neuron can be transmitted more easily to farther and more neurons. This can increase the speed of the collective dynamics and cause a reduction in the barrier height. For a certain ϕ, when refractory periods are greater than 1, the threshold $B_c(\phi)$ remains at the same values but the barrier distribution becomes less abrupt.

The study of the avalanche is crucial for investigating the critical features of complex systems. Similar to those used in

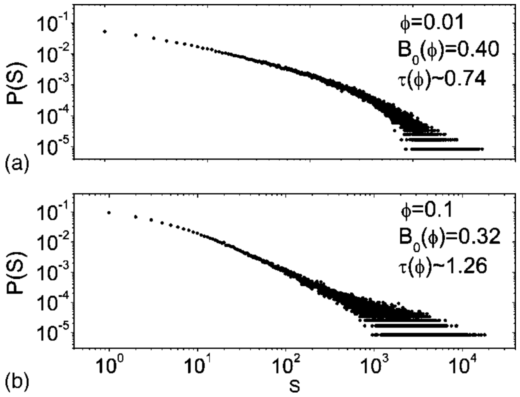

FIG. 2. Distribution of the $B_0(\phi)$ avalanche temporal sizes with system $L=1000$, $k=2$ and absolute refractory period $T_r=1$: ϕ = (*a*) 0.01; (b) 0.1.

Refs. [8] and [9], for a certain ϕ, we present the definition of the $B_0(\phi)$ avalanche, where $B_0(\phi)$ $[0<B_0(\phi)<B_c(\phi)]$ is an auxiliary parameter used to define the avalanche. Suppose that at time s, the smallest random number in the system is larger than $B_0(\phi)$. According to the rules of the model, if, at time step $s+1$, the lowest of the new random numbers selected is less than $B_0(\phi)$, a $B_0(\phi)$ avalanche begins. The avalanche continues to run if the lowest random barrier whose corresponding neuron can be fired is less than $B_0(\phi)$. The avalanche stops, say at time $s+S$, when the lowest number is larger than $B_0(\phi)$ for the first time. The $B_0(\phi)$ avalanche size is defined as the duration of the avalanche S.

In Fig. 2, we draw the probability distribution $P(S)$ of $B_0(\phi)$ avalanches of size S for $L=1000$, $k=2$, and $T_r=1$ when $\phi=0.01$ and 0.1, respectively. We find that $B_0(\phi)$ avalanche distributions obey power-law behavior $P(S)\propto S^{-\tau(\phi)}$ in our model. With the increment of ϕ, the exponent $\tau(\phi)$ increases. This phenomenon may be caused by the increasing of randomness in our model.

At the same time, in Fig. 3, we demonstrate the dependence of the exponent τ for the $B_0(\phi)$ avalanche distribution on the refractory period T_r for the system with $L=1000$, $k=2$, $\phi=0.01$, and $B_0(\phi)=0.40$. As T_r increases, the exponent τ increases. Different refractory periods and consequently different exponents in the distribution for avalanches could characterize different time scale features of the brain: short, rapidly adaptive ones (as, for example, breath control) and slower long-term ones (say, for example, language) [7].

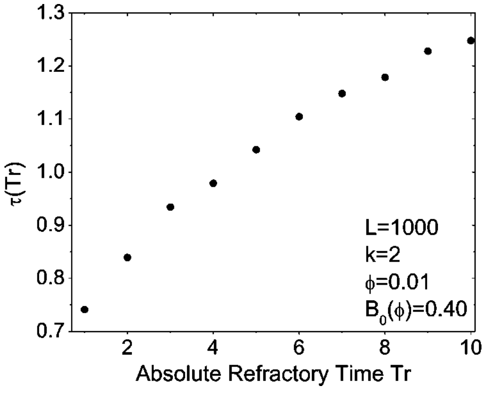

FIG. 3. Dependence of the exponent τ of the power law in Fig. 2(a) as a function of the refractory period T_r.

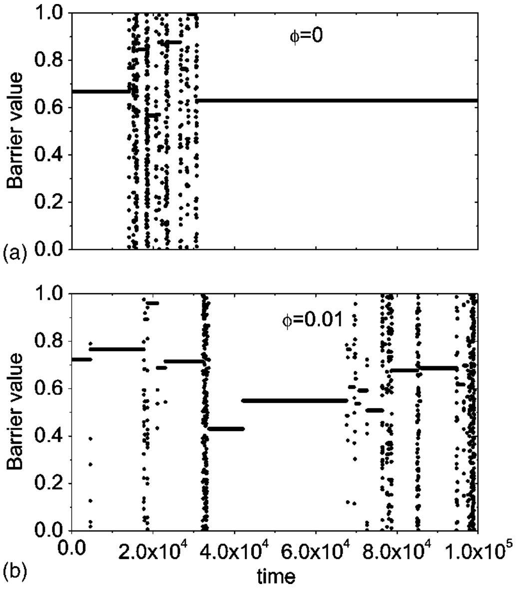

FIG. 4. Temporal dependence of the value of a barrier at the maximally connected site for ϕ= (*a*) 0 and (b) 0.01 for $L=1000$, $k=2$, and $T_r=1$, when the system is at the self-organized critical state.

In the critical state, each barrier suffers bursts of activity alternating with long periods of calm. One way to characterize this intermittency is observing the activity of a single barrier. In Fig. 4, we present the temporal dependence of the value of a single barrier at the maximally connected site for $\phi=0$ and 0.01, respectively, during a time interval when the system is at the critical state. They both exhibit "punctuated equilibrium" behaviors. But we find a drastic reduction in the periods of calm from the order topology to the small-world networks. With the increment of ϕ, the number of links of the maximally connected site increases. The range of local interaction at the maximally connected site enlarges, and then there are more chances to change its stability by changing its barrier value. So the periods of calm in Fig. 4(b) become shorter and the density of black points becomes greater than that in Fig. 4(a). The model exhibits "intermittent dynamics" which resemble the measured results of the firing response of a single neuron in a monkey visual cortex [10].

We investigate the temporal correlation between the minimum barriers and focus on two quantities: the probability distribution of first return times and distribution of all return times. Define the size t of the first return time as the number of consecutive time steps during which the observed barrier remains constant. Note that this characterization allows for an easier comparison with experiments; the magnitudes more often measured in experiments with actual neurons are the interspike time periods (in our vocabulary, first return times). In Fig. 5, we show the probability distribution of first return times when the system is at the critical state for $\phi=0.01$; the calculation was done above for $L=1000$, $k=2$, and $T_r=1$. The distribution satisfies power-law behavior $P(t)\propto t^{-\tau_f}$ with

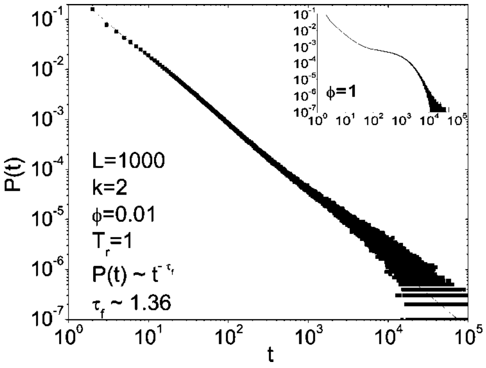

FIG. 5. The probability distribution of first return times for system size $L=1000$, $k=2$, $\phi=0.01$, and the absolute refractory period $T_r=1$.

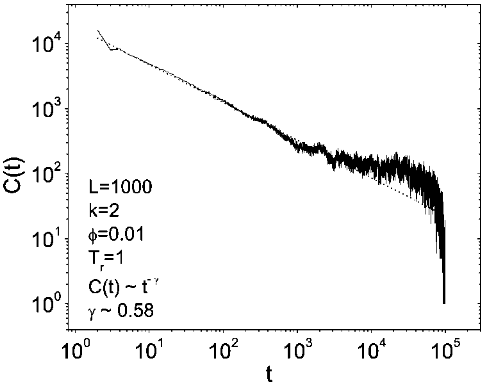

FIG. 6. Distribution of all return times for system size $L=1000$, $k=2$, $\phi=0.01$, and the absolute refractory period $T_r=1$.

exponent $\tau_f \approx 1.36$. The exponent for the first return time probability distribution is different from the one for the regular model with $k=1$, $T_r=1$ ($\tau_f=1.60\pm0.04$) [7] or the regular model with $k=2$, $T_r=1$ ($\tau_f=1.54\pm0.04$). But with the increment of ϕ the probability distributions of first return times do not obey a power law. The same plot for $\phi=1$ is shown in the inset of Fig. 5 and we can see that the plot does not display the power-law behavior. Our result here is different from that in the random neighbor (RN) model [11]. It may be caused by the different randomness. The randomness in the RN model is in fact a kind of "annealed" randomness [11], but the randomness in our model is "quenched," that is, the spatial structure of the network is fixed. In fact, Papa and da Silva have found not only power-law but also non-power-law types of distribution by analyzing experimental data of real interspike measurements in the visual cortex of macaques [12].

Another quantity is the distribution of all return times distribution $C(t)$. We define the size t of all return times as the number of time steps, that is, if a given neuron fires (with the minimum barrier) at step t_0, it will fire at t_0+t regardless of what happens at intermediate steps. The distribution of all return times in Fig. 6 also decays slowly via a power law $C(t) \propto t^{-\gamma}$, $\gamma \sim 0.58$. Since $C(t)$ is the autocorrelation function of the activity, the power spectrum is very simple [9]. According to the Wiener-Khinchin theorem [13], $S(f)$ is the Fourier transform of the (auto)correlation function:

$$S(f) = \int_{-\infty}^{+\infty} C(t) e^{2\pi i f t} dt. \tag{1}$$

We measure the power spectrum $S(f)$ of $C(t)$ for $L=1000$, $k=2$, $T_r=1$, and $\phi=0.01$ as shown in Fig. 7. The exponent α in the power spectrum $S(f) \propto 1/f^{\alpha}$ is $\alpha \sim 0.55$. So it can be categorized as $1/f$ type noise. Therefore, the system exhibits self-organized criticality. $1/f$ noise is the characteristic signature of fluctuation on many time scales. It has been observed in the pulse trains of nerve cells belonging to various brain structures, such as the auditory nerve and the mesencephalic reticular formation [14].

IV. CONCLUSION

In this paper, we provide a one-dimensional lattice system with small-world structure to investigate scale invariance in the activity of neural populations. From a neuron-biological view, the network structure of our model is closer to the fact than that in Ref. [7]. In the process of simulation, our model displays power-law behavior of avalanche size and two long-range temporal correlations for first return times and all return times and $1/f$ noise. At the same time, we find that there are different avalanche dynamical behaviors for different topologies of the network. However, our model is very simple for brain functioning and neglects the details of the elements in the brain. We have only simulated the simplest characteristic of the neuron. Models for more complex neural behaviors should be tested and developed in future research.

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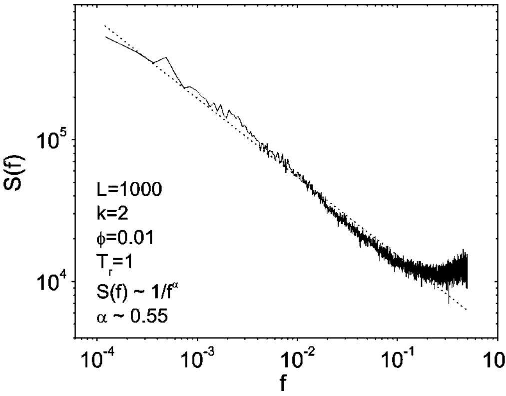

FIG. 7. Power spectrum for system size $L=1000$, $k=2$, $\phi=0.01$, and the absolute refractory period $T_r=1$.

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